

# 6

## Cotangent Bundles

In many mechanics problems, the phase space is the cotangent bundle  $T^*Q$  of a configuration space  $Q$ . There is an “intrinsic” symplectic structure on  $T^*Q$  that can be described in various equivalent ways. Assume first that  $Q$  is  $n$ -dimensional, and pick local coordinates  $(q^1, \dots, q^n)$  on  $Q$ . Since  $(dq^1, \dots, dq^n)$  is a basis of  $T_q^*Q$ , we can write any  $\alpha \in T_q^*Q$  as  $\alpha = p_i dq^i$ . Then  $(q^1, \dots, q^n, p_1, \dots, p_n)$  are local coordinates on  $T^*Q$ . Define the **canonical symplectic form** on  $T^*Q$  by

$$\Omega = dq^i \wedge dp_i.$$

This defines a closed two-form  $\Omega$  that can be checked to be independent of the choice of coordinates  $(q^1, \dots, q^n)$ . Observe that  $\Omega$  is locally constant, that is, is independent of the base point  $(q^1, \dots, q^n, p_1, \dots, p_n)$ . In this section we show how to do this construction intrinsically and we will study this canonical symplectic structure in some detail.

### 6.1 The Linear Case

To motivate a coordinate independent definition of  $\Omega$ , consider the case in which  $Q$  is a vector space  $W$  (which could be infinite dimensional), so that  $T^*Q = W \times W^*$ . We have already described the canonical two-form on  $W \times W^*$ :

$$\Omega_{(w, \alpha)}((u, \beta), (v, \gamma)) = \langle \gamma, u \rangle - \langle \beta, v \rangle, \quad (6.1.1)$$

where  $(w, \alpha) \in W \times W^*$  is the base point,  $u, v \in W$ , and  $\beta, \gamma \in W^*$ . This canonical two-form will be constructed from the **canonical one-form**  $\Theta$ , defined as follows:

$$\Theta_{(w, \alpha)}(u, \beta) = \langle \alpha, u \rangle. \quad (6.1.2)$$

The next proposition shows that the canonical two-form (6.1.1) is exact:

$$\Omega = -\mathbf{d}\Theta. \quad (6.1.3)$$

We begin with a computation that reconciles these formulas with their coordinate expressions.

**Proposition 6.1.1.** *In the finite-dimensional case the symplectic form  $\Omega$  defined by (6.1.1) can be written  $\Omega = dq^i \wedge dp_i$  in coordinates  $q^1, \dots, q^n$  on  $W$  and corresponding dual coordinates  $p_1, \dots, p_n$  on  $W^*$ . The associated canonical one-form is given by  $\Theta = p_i dq^i$  and (6.1.3) holds.*

**Proof.** If  $(q^1, \dots, q^n, p_1, \dots, p_n)$  are coordinates on  $T^*W$  then

$$\left( \frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}, \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n} \right)$$

denotes the induced basis for  $T_{(w, \alpha)}(T^*W)$ , and  $(dq^1, \dots, dq^n, dp_1, \dots, dp_n)$  denotes the associated dual basis of  $T_{(w, \alpha)}^*(T^*W)$ . Write

$$(u, \beta) = \left( u^j \frac{\partial}{\partial q^j}, \beta_j \frac{\partial}{\partial p_j} \right)$$

and similarly for  $(v, \gamma)$ . Hence

$$\begin{aligned} (dq^i \wedge dp_i)_{(w, \alpha)}((u, \beta), (v, \gamma)) &= (dq^i \otimes dp_i - dp_i \otimes dq^i)((u, \beta), (v, \gamma)) \\ &= dq^i(u, \beta) dp_i(v, \gamma) - dp_i(u, \beta) dq^i(v, \gamma) \\ &= u^i \gamma_i - \beta_i v^i. \end{aligned}$$

Also,  $\Omega_{(w, \alpha)}((u, \beta), (v, \gamma)) = \gamma(u) - \beta(v) = \gamma_i u^i - \beta_i v^i$ . Thus,

$$\Omega = dq^i \wedge dp_i.$$

Similarly,

$$(p_i dq^i)_{(w, \alpha)}(u, \beta) = \alpha_i dq^i(u, \beta) = \alpha_i u^i,$$

and

$$\Theta_{(w, \alpha)}(u, \beta) = \alpha(u) = \alpha_i u^i.$$

Comparing, we get  $\Theta = p_i dq^i$ . Therefore,

$$-\mathbf{d}\Theta = -\mathbf{d}(p_i dq^i) = dq^i \wedge dp_i = \Omega. \quad \blacksquare$$

To verify (6.1.3) for the infinite-dimensional case, use (6.1.2) and the second formula 6 of the table at the end of §4.4 to give

$$\begin{aligned} \mathbf{d}\Theta_{(w,\alpha)}((u_1, \beta_1), (u_2, \beta_2)) &= [\mathbf{D}\Theta_{(w,\alpha)} \cdot (u_1, \beta_1)] \cdot (u_2, \beta_2) \\ &\quad - [\mathbf{D}\Theta_{(w,\alpha)} \cdot (u_2, \beta_2)] \cdot (u_1, \beta_1) \\ &= \langle \beta_1, u_2 \rangle - \langle \beta_2, u_1 \rangle, \end{aligned}$$

since  $\mathbf{D}\Theta_{(w,\alpha)} \cdot (u, \beta) = \langle \beta, \cdot \rangle$ . But this equals  $-\Omega_{(w,\alpha)}((u_1, \beta_1), (u_2, \beta_2))$ .

To give an intrinsic interpretation to  $\Theta$ , let us prove that

$$\Theta_{(w,\alpha)} \cdot (u, \beta) = \langle \alpha, T_{(w,\alpha)}\pi_W(u, \beta) \rangle, \quad (6.1.4)$$

where  $\pi_W : W \times W^* \rightarrow W$  is the projection. Indeed (6.1.4) coincides with (6.1.2) since  $T_{(w,\alpha)}\pi_W : W \times W^* \rightarrow W$  is the projection on the first factor.

## Exercises

- ◇ **Exercise 6.1-1 (Jacobi–Haretu Coordinates).** Consider the configuration space  $Q = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$  with elements denoted  $\mathbf{r}_1, \mathbf{r}_2$ , and  $\mathbf{r}_3$ . Call the conjugate momenta  $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$  and equip the phase space  $T^*Q$  with the canonical symplectic structure  $\Omega$ . Let  $\mathbf{j} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$ . Let  $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$  and let  $\mathbf{s} = \mathbf{r}_3 - \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$ . Show that the form  $\Omega$  pulled back to the level sets of  $\mathbf{j}$  has the form  $\Omega = d\mathbf{r} \wedge d\pi + d\mathbf{s} \wedge d\sigma$ , where the variables  $\pi$  and  $\sigma$  are defined by  $\pi = \frac{1}{2}(\mathbf{p}_2 - \mathbf{p}_1)$  and  $\sigma = \mathbf{p}_3$ .

## 6.2 The Nonlinear Case

**Definition 6.2.1.** Let  $Q$  be a manifold. We define  $\Omega = -\mathbf{d}\Theta$ , where  $\Theta$  is the one-form on  $T^*Q$  defined analogous to (6.1.4), namely

$$\Theta_\beta(v) = \langle \beta, T\pi_Q \cdot v \rangle, \quad (6.2.1)$$

where  $\beta \in T^*Q$ ,  $v \in T_\beta(T^*Q)$ ,  $\pi_Q : T^*Q \rightarrow Q$  is the projection, and  $T\pi_Q : T(T^*Q) \rightarrow TQ$  is the tangent map of  $\pi_Q$ .

The computations in Proposition 6.1.1 show that  $(T^*Q, \Omega = -\mathbf{d}\Theta)$  is a symplectic manifold; indeed, in local coordinates with  $(w, \alpha) \in U \times W^*$ , where  $U$  is open in  $W$ , and where  $(u, \beta), (v, \gamma) \in W \times W^*$ , the two-form  $\Omega = -\mathbf{d}\Theta$  is given by

$$\Omega_{(w,\alpha)}((u, \beta), (v, \gamma)) = \gamma(u) - \beta(v). \quad (6.2.2)$$

Darboux's theorem and its corollary can be interpreted as asserting that any (strong) symplectic manifold locally looks like  $W \times W^*$  in suitable local coordinates.

**Hamiltonian Vector Fields.** For a function  $H : T^*Q \rightarrow \mathbb{R}$ , the Hamiltonian vector field  $X_H$  on the cotangent bundle  $T^*Q$  is given in canonical cotangent bundle charts  $U \times W^*$ , where  $U$  is open in  $W$ , by

$$X_H(w, \alpha) = \left( \frac{\delta H}{\delta \alpha}, -\frac{\delta H}{\delta w} \right). \quad (6.2.3)$$

Indeed, denoting  $X_H(w, \alpha) = (w, \alpha, v, \gamma)$ , for any  $(u, \beta) \in W \times W^*$  we have

$$\begin{aligned} \mathbf{d}H_{(w, \alpha)} \cdot (u, \beta) &= \mathbf{D}_w H_{(w, \alpha)} \cdot u + \mathbf{D}_\alpha H_{(w, \alpha)} \cdot \beta \\ &= \left\langle \frac{\delta H}{\delta w}, u \right\rangle + \left\langle \beta, \frac{\delta H}{\delta \alpha} \right\rangle \end{aligned} \quad (6.2.4)$$

which, by definition and (6.2.2), equals

$$\Omega_{(w, \alpha)}(X_H(w, \alpha), (u, \beta)) = \langle \beta, v \rangle - \langle \gamma, u \rangle. \quad (6.2.5)$$

Comparing (6.2.4) and (6.2.5) gives (6.2.3). In finite dimensions, (6.2.3) is the familiar right-hand side of Hamilton's equations.

**Poisson Brackets.** Formula (6.2.3) and the definition of the Poisson bracket show that in canonical cotangent bundle charts,

$$\{f, g\}(w, \alpha) = \left\langle \frac{\delta f}{\delta w}, \frac{\delta g}{\delta \alpha} \right\rangle - \left\langle \frac{\delta g}{\delta w}, \frac{\delta f}{\delta \alpha} \right\rangle, \quad (6.2.6)$$

which in finite dimensions becomes

$$\{f, g\}(q^i, p_i) = \sum_{i=1}^n \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right). \quad (6.2.7)$$

**Pull Back Characterization.** Another characterization of the canonical one-form that is sometimes useful is the following:

**Proposition 6.2.2.**  $\Theta$  is the unique one-form on  $T^*Q$  such that

$$\alpha^* \Theta = \alpha \quad (6.2.8)$$

for any local one-form  $\alpha$  on  $Q$ , where, on the left-hand side,  $\alpha$  is regarded as a map (of some open subset of)  $Q$  to  $T^*Q$ .

**Proof.** In finite dimensions, if  $\alpha = \alpha_i(q^j) dq^i$  and  $\Theta = p_i dq^i$ , then to calculate  $\alpha^* \Theta$  means that we substitute  $p_i = \alpha_i(q^j)$  into  $\Theta$ , a process which clearly gives back  $\alpha$ , so  $\alpha^* \Theta = \alpha$ . The general argument is as follows. If  $\Theta$  is the canonical one-form on  $T^*Q$ , and  $v \in T_q Q$ , then

$$\begin{aligned} (\alpha^* \Theta)_q \cdot v &= \Theta_{\alpha(q)} \cdot T_q \alpha(v) = \langle \alpha(q), T_{\alpha(q)} \pi_Q(T_q \alpha(v)) \rangle \\ &= \langle \alpha(q), T_q(\pi_Q \circ \alpha)(v) \rangle = \alpha(q) \cdot v \end{aligned}$$

since  $\pi_Q \circ \alpha = \text{identity on } Q$ .

For the converse, assume that  $\Theta$  is a one-form on  $T^*Q$  satisfying (6.2.8). We will show that it must then be the canonical one-form (6.2.1). In finite dimensions this is straightforward: if  $\Theta = A_i dq^i + B^i dp_i$  for  $A_i, B^i$  functions of  $(q^j, p_j)$ , then

$$\alpha^* \Theta = (A_i \circ \alpha) dq^i + (B^i \circ \alpha) d\alpha_i = \left( A_j \circ \alpha + (B^i \circ \alpha) \frac{\partial \alpha_i}{\partial q^j} \right) dq^j$$

which equals  $\alpha = \alpha_i dq^i$  iff

$$A_j \circ \alpha + (B^i \circ \alpha) \frac{\partial \alpha_i}{\partial q^j} = \alpha_j.$$

Since this must hold for all  $\alpha_j$ , putting  $\alpha_1, \dots, \alpha_n$  constant it follows that  $A_j \circ \alpha = \alpha_j$ , that is,  $A_j = p_j$ . Therefore, the remaining equation is

$$(B^i \circ \alpha) \partial \alpha_i / \partial q^j = 0$$

for any  $\alpha_i$ ; choosing  $\alpha_i(q^1, \dots, q^n) = q_0^i + (q^i - q_0^i)p_i^0$  (no sum) implies  $0 = (B^j \circ \alpha)(q_0^1, \dots, q_0^n) = p_j^0$  for all  $(q_0^j, p_j^0)$ , that is,  $B^j = 0$  and thus  $\Theta = p_i dq^i$ .<sup>1</sup> ■

## Exercises

- ◇ **Exercise 6.2-1.** Let  $N$  be a submanifold of  $M$  and denote by  $\Theta_N$  and  $\Theta_M$  the canonical one-forms on the cotangent bundles  $\pi_N : T^*N \rightarrow N$  and  $\pi_M : T^*M \rightarrow M$ , respectively. Let  $\pi : (T^*M)|_N \rightarrow T^*N$  be the projection defined by  $\pi(\alpha_n) = \alpha_n|_{T_n N}$ , where  $n \in N$  and  $\alpha_n \in T_n^*M$ . Show that  $\pi^* \Theta_N = i^* \Theta_M$ , where  $i : (T^*M)|_N \rightarrow T^*M$  is the inclusion.
- ◇ **Exercise 6.2-2.** Let  $f : Q \rightarrow \mathbb{R}$  and  $X \in \mathfrak{X}(T^*Q)$ . Show that

$$\Theta(X) \circ \mathbf{d}f = X[f \circ \pi_Q] \circ \mathbf{d}f.$$

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<sup>1</sup>In infinite dimensions, the proof is slightly different. We will show that if (6.2.8) holds then  $\Theta$  is locally given by (6.1.4) and thus it is the canonical one-form. If  $U \subset E$  is the chart domain the Banach space  $E$  modeling  $Q$  for any  $v \in E$  we have

$$(\alpha^* \Theta)_u \cdot (u, v) = \Theta(u, \alpha(u)) \cdot (v, \mathbf{D}\alpha(u) \cdot v),$$

where  $\alpha$  is given locally by  $u \mapsto (u, \alpha(u))$  for  $\alpha : U \rightarrow E^*$ . Thus (6.2.8) is equivalent to

$$\Theta_{(u, \alpha(u))} \cdot (v, \mathbf{D}\alpha(u) \cdot v) = \langle \alpha(u), v \rangle$$

which would imply (6.1.4) and hence  $\Theta$  being the canonical one-form, provided we can show that for prescribed  $\gamma, \delta \in E^*, u \in U, v \in E$  there is an  $\alpha : U \rightarrow E^*$  such that  $\alpha(u) = \gamma, \mathbf{D}\alpha(u) \cdot v = \delta$ . Such a mapping is constructed in the following way. For  $v = 0$  choose  $\alpha(u)$  to equal  $\gamma$  for all  $u$ . For  $v \neq 0$ , by the Hahn-Banach theorem one can find a  $\varphi \in E^*$  such that  $\varphi(v) = 1$ . Now set  $\alpha(x) = \gamma - \varphi(u)\delta + \varphi(x)\delta$ .]

- ◇ **Exercise 6.2-3.** Let  $Q$  be a given configuration manifold and let the *extended phase space* be defined by  $(T^*Q) \times \mathbb{R}$ . Given a time dependent vector field  $X$  on  $T^*Q$ , extend it to a vector field  $\tilde{X}$  on  $(T^*Q) \times \mathbb{R}$  by  $\tilde{X} = (X, 1)$ .

Let  $H$  be a (possibly time-dependent) function on  $(T^*Q) \times \mathbb{R}$  and set

$$\Omega_H = \Omega + dH \wedge dt,$$

where  $\Omega$  is the canonical two-form. Show that  $X$  is the Hamiltonian vector field for  $H$  if and only if

$$\mathbf{i}_{\tilde{X}}\Omega_H = 0.$$

- ◇ **Exercise 6.2-4.** Give an example of a symplectic manifold  $(P, \Omega)$ , where  $\Omega$  is exact, but  $P$  is *not* a cotangent bundle.

## 6.3 Cotangent Lifts

We now describe an important way to create symplectic transformations on cotangent bundles.

**Definition 6.3.1.** Given two manifolds  $Q$  and  $S$  and a diffeomorphism  $f : Q \rightarrow S$ , the *cotangent lift*  $T^*f : T^*S \rightarrow T^*Q$  of  $f$  is defined by

$$\langle T^*f(\alpha_s), v \rangle = \langle \alpha_s, (Tf \cdot v) \rangle, \quad (6.3.1)$$

where

$$\alpha_s \in T_s^*S, \quad v \in T_qQ, \quad \text{and} \quad s = f(q).$$

The importance of this construction is that  $T^*f$  is guaranteed to be symplectic; it is often called a “point transformation” because it arises from a diffeomorphism on points in configuration space. Notice that while  $Tf$  covers  $f$ ,  $T^*f$  covers  $f^{-1}$ . Denote by  $\pi_Q : T^*Q \rightarrow Q$  and  $\pi_S : T^*S \rightarrow S$ , the canonical cotangent bundle projections.

**Proposition 6.3.2.** A diffeomorphism  $\varphi : T^*S \rightarrow T^*Q$  preserves the canonical one-forms  $\Theta_Q$  and  $\Theta_S$  on  $T^*Q$  and  $T^*S$ , respectively, if and only if  $\varphi$  is the cotangent lift  $T^*f$  of some diffeomorphism  $f : Q \rightarrow S$ .

**Proof.** First assume that  $f : Q \rightarrow S$  is a diffeomorphism. Then for arbitrary  $\beta \in T^*S$  and  $v \in T_\beta(T^*S)$ , we have

$$\begin{aligned} ((T^*f)^*\Theta_Q)_\beta \cdot v &= (\Theta_Q)_{T^*f(\beta)} \cdot TT^*f(v) \\ &= \langle T^*f(\beta), (T\pi_Q \circ TT^*f) \cdot v \rangle \\ &= \langle \beta, T(f \circ \pi_Q \circ T^*f) \cdot v \rangle \\ &= \langle \beta, T\pi_S \cdot v \rangle = \Theta_{S\beta} \cdot v, \end{aligned}$$

since  $f \circ \pi_Q \circ T^*f = \pi_S$ . ■

Conversely, assume that  $\varphi^*\Theta_Q = \Theta_S$ , that is,

$$\langle \varphi(\beta), T(\pi_Q \circ \varphi)(v) \rangle = \langle \beta, T\pi_S(v) \rangle \quad (6.3.2)$$

for all  $\beta \in T^*S$  and  $v \in T_\beta(T^*S)$ . Since  $\varphi$  is a diffeomorphism, the range of  $T_\beta(\pi_Q \circ \varphi)$  is  $T_{\pi_Q(\varphi(\beta))}Q$ , so that letting  $\beta = 0$  in (6.3.2) implies that  $\varphi(0) = 0$ . Arguing similarly for  $\varphi^{-1}$  instead of  $\varphi$ , we conclude that  $\varphi$  restricted to the zero section  $S$  of  $T^*S$  is a diffeomorphism onto the zero section  $Q$  of  $T^*Q$ . Define  $f : Q \rightarrow S$  by  $f = \varphi^{-1}|_Q$ . We will show below that  $\varphi$  is fiber-preserving or, equivalently, that  $f \circ \pi_Q = \pi_S \circ \varphi^{-1}$ . For this we need the following:

**Lemma 6.3.3.** *Define the flow  $F_t^Q$  on  $T^*Q$  by  $F_t^Q(\alpha) = e^t\alpha$  and let  $V_Q$  be the vector field it generates. Then*

$$\langle \Theta_Q, V_Q \rangle = 0, \quad \mathcal{L}_{V_Q}\Theta_Q = \Theta_Q, \quad \text{and} \quad \mathbf{i}_{V_Q}\Omega_Q = -\Theta_Q. \quad (6.3.3)$$

**Proof.** Since  $F_t^Q$  is fiber-preserving,  $V_Q$  will be tangent to the fibers and hence  $T\pi_Q \circ V_Q = 0$ . This implies by (6.2.1) that  $\langle \Theta_Q, V_Q \rangle = 0$ . To prove the second formula, note that  $\pi_Q \circ F_t^Q = \pi_Q$ . Let  $\alpha \in T_q^*Q$ ,  $v \in T_\alpha(T^*Q)$ , and  $\Theta_\alpha$  denote  $\Theta_Q$  evaluated at  $\alpha$ . We have

$$\begin{aligned} ((F_t^Q)^*\Theta)_\alpha \cdot v &= \Theta_{F_t^Q(\alpha)} \cdot TF_t^Q(v) \\ &= \left\langle F_t^Q(\alpha), (T\pi_Q \circ TF_t^Q)(v) \right\rangle \\ &= \left\langle e^t\alpha, T(\pi_Q \circ F_t^Q)(v) \right\rangle \\ &= e^t \langle \alpha, T\pi_Q(v) \rangle = e^t \Theta_\alpha \cdot v, \end{aligned}$$

that is,

$$(F_t^Q)^*\Theta_Q = e^t\Theta_Q.$$

Taking the derivative relative to  $t$  at  $t = 0$  yields the second formula. Finally, the first two formulas imply

$$\mathbf{i}_{V_Q}\Omega_Q = -\mathbf{i}_{V_Q}\mathbf{d}\Theta_Q = -\mathcal{L}_{V_Q}\Theta_Q + \mathbf{d}\mathbf{i}_{V_Q}\Theta_Q = -\Theta_Q. \quad \blacktriangledown$$

Continuing the proof of the proposition, note that by (6.3.3) we have

$$\begin{aligned} \mathbf{i}_{\varphi^*V_Q}\Omega_S &= \mathbf{i}_{\varphi^*V_Q}\varphi^*\Omega_Q = \varphi^*(\mathbf{i}_{V_Q}\Omega_Q) \\ &= -\varphi^*\Theta_Q = -\Theta_S = \mathbf{i}_{V_S}\Omega_S, \end{aligned}$$

so that weak nondegeneracy of  $\Omega_S$  implies  $\varphi^*V_Q = V_S$ . Thus  $\varphi$  commutes with the flows  $F_t^Q$  and  $F_t^S$ , that is, for any  $\beta \in T^*S$  we have  $\varphi(e^t\beta) = e^t\varphi(\beta)$ . Letting  $t \rightarrow -\infty$  in this equality implies  $(\varphi \circ \pi_S)(\beta) = (\pi_Q \circ \varphi)(\beta)$  since  $e^t\beta \rightarrow \pi_S(\beta)$  and  $e^t\varphi(\beta) \rightarrow (\pi_Q \circ \varphi)(\beta)$  for  $t \rightarrow -\infty$ . Thus  $\pi_Q \circ \varphi = \varphi \circ \pi_S$ , or  $f \circ \pi_Q = \pi_S \circ \varphi^{-1}$ .

Finally, we show that  $T^*f = \varphi$ . For  $\beta \in T^*S$ ,  $v \in T_\beta(T^*S)$ , (6.3.2) gives

$$\begin{aligned}\langle T^*f(\beta), T(\pi_Q \circ \varphi)(v) \rangle &= \langle \beta, T(f \circ \pi_Q \circ \varphi)(v) \rangle \\ &= \langle \beta, T\pi_S(v) \rangle = (\Theta_S)_\beta \cdot v \\ &= (\varphi^*\Theta_Q)_\beta \cdot v = (\Theta_Q)_{\varphi(\beta)} \cdot T_\beta\varphi(v) \\ &= \langle \varphi(\beta), T_\beta(\pi_Q \circ \varphi)(v) \rangle,\end{aligned}$$

which shows that  $T^*f = \varphi$  since the range of  $T_\beta(\pi_Q \circ \varphi)$  is the whole tangent space at  $(\pi_Q \circ \varphi)(\beta)$  to  $Q$ . ■

In finite dimensions, the first part of this proposition can be seen in coordinates as follows. Write  $(s^1, \dots, s^n) = f(q^1, \dots, q^n)$  and define

$$p_j = \frac{\partial s^i}{\partial q^j} r_i, \quad (6.3.4)$$

where  $(q^1, \dots, q^n, p_1, \dots, p_n)$  are cotangent bundle coordinates on  $T^*Q$  and  $(s^1, \dots, s^n, r_1, \dots, r_n)$  on  $T^*S$ . Since  $f$  is a diffeomorphism, it determines the  $q^i$  in terms of the  $s^j$ , say  $q^i = q^i(s^1, \dots, s^n)$ , so both  $q^i$  and  $p_j$  are functions of  $(s^1, \dots, s^n, r_1, \dots, r_n)$ . The map  $T^*f$  is given by

$$(s^1, \dots, s^n, r_1, \dots, r_n) \mapsto (q^1, \dots, q^n, p_1, \dots, p_n). \quad (6.3.5)$$

To see that (6.3.5) preserves the canonical one-form, use the chain rule and (6.3.4):

$$r_i ds^i = r_i \frac{\partial s^i}{\partial q^k} dq^k = p_k dq^k. \quad (6.3.6)$$

Note that if  $f$  and  $g$  are diffeomorphisms of  $Q$ , then

$$T^*(f \circ g) = T^*g \circ T^*f, \quad (6.3.7)$$

that is, the cotangent lift switches the order of compositions; in fact, *it is useful to think of  $T^*f$  as the **adjoint** of  $Tf$* .

## Exercises

- ◇ **Exercise 6.3-1.** *The Lorentz group*  $\mathcal{L}$  is the group of invertible linear transformations of  $\mathbb{R}^4$  to itself that preserve the quadratic form  $x^2 + y^2 + z^2 - c^2 t^2$ , where  $c$  is a constant, the speed of light. Describe all elements of this group. Let  $\Lambda_0$  denote one of these transformations. Map  $\mathcal{L}$  to itself by  $\Lambda \mapsto \Lambda_0 \Lambda$ . Calculate the cotangent lift of this map.
- ◇ **Exercise 6.3-2.** We have shown that a transformation of  $T^*Q$  is the cotangent lift of a diffeomorphism of configuration space if and only if it preserves the canonical one-form. Find this result in Whittaker's book.

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## 6.4 Lifts of Actions

A **left action** of a group  $G$  on a manifold  $M$  associates to each group element  $g \in G$  a diffeomorphism  $\Phi_g$  of  $M$ , such that  $\Phi_{gh} = \Phi_g \circ \Phi_h$ . Thus, the collection of  $\Phi_g$ 's is a *group of transformations of  $M$* . If we replace the condition  $\Phi_{gh} = \Phi_g \circ \Phi_h$  by  $\Psi_{gh} = \Psi_h \circ \Psi_g$  we speak of a **right action**. We often write  $\Phi_g(m) = g \cdot m$  and  $\Psi_g(m) = m \cdot g$  for  $m \in M$ .

**Definition 6.4.1.** *Let  $\Phi$  be an action of a group  $G$  on a manifold  $Q$ . The **right lift**  $\Phi^*$  of the action  $\Phi$  to the symplectic manifold  $T^*Q$  is the right action defined by the rule*

$$\Phi_g^*(\alpha) = (T_{g^{-1} \cdot q}^* \Phi_g)(\alpha), \quad (6.4.1)$$

where  $g \in G$ ,  $\alpha \in T_q^*Q$ , and  $T^*\Phi_g$  is the cotangent lift of the diffeomorphism  $\Phi_g : Q \rightarrow Q$ .

By (6.3.7), we see that

$$\Phi_{gh}^* = T^*\Phi_{gh} = T^*(\Phi_g \circ \Phi_h) = T^*\Phi_h \circ T^*\Phi_g = \Phi_h^* \circ \Phi_g^* \quad (6.4.2)$$

so  $\Phi^*$  is a right action. To get a **left action**, denoted  $\Phi_*$  and called the **left lift** of  $\Phi$ , one sets

$$(\Phi_*)_g = T_{g \cdot q}^*(\Phi_{g^{-1}}). \quad (6.4.3)$$

In either case these lifted actions are actions by canonical transformations because of Proposition 6.3.2. We shall return to the study of actions of groups after we study Lie groups in Chapter 9.

### Examples

(a) For a system of  $N$  particles in  $\mathbb{R}^3$ , we choose the configuration space  $Q = \mathbb{R}^{3N}$ . We write  $(\mathbf{q}_j)$  for an  $N$ -tuple of vectors labeled by  $j = 1, \dots, N$ . Similarly, elements of the momentum phase space  $P = T^*\mathbb{R}^{3N} \cong \mathbb{R}^{6N} \cong \mathbb{R}^{3N} \times \mathbb{R}^{3N}$  are denoted  $(\mathbf{q}_j, \mathbf{p}^j)$ . Let the additive group  $G = \mathbb{R}^3$  of translations act on  $Q$  according to

$$\Phi_{\mathbf{x}}(\mathbf{q}_j) = \mathbf{q}_j + \mathbf{x}, \quad \text{where } \mathbf{x} \in \mathbb{R}^3. \quad (6.4.4)$$

Each of the  $N$  position vectors  $\mathbf{q}_j$  is translated by the same vector  $\mathbf{x}$ .

Lifting the diffeomorphism  $\Phi_{\mathbf{x}} : Q \rightarrow Q$ , we obtain an action  $\Phi^*$  of  $G$  on  $P$ . We assert that

$$\Phi_{\mathbf{x}}^*(\mathbf{q}_j, \mathbf{p}^j) = (\mathbf{q}_j - \mathbf{x}, \mathbf{p}^j). \quad (6.4.5)$$

To verify (6.4.5), observe that  $T\Phi_{\mathbf{x}} : TQ \rightarrow TQ$  is given by

$$(\mathbf{q}_i, \dot{\mathbf{q}}_j) \mapsto (\mathbf{q}_i + \mathbf{x}, \dot{\mathbf{q}}_j) \quad (6.4.6)$$

so its dual is  $(\mathbf{q}_i, \mathbf{p}^j) \mapsto (\mathbf{q}_i - \mathbf{x}, \mathbf{p}^j)$ . ♦

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(b) Consider the action of  $\mathrm{GL}(n, \mathbb{R})$ , the group of  $n \times n$  invertible matrices, or more properly, the group of invertible linear transformations of  $\mathbb{R}^n$  to itself, on  $\mathbb{R}^n$  given by

$$\Phi_A(\mathbf{q}) = A\mathbf{q}. \quad (6.4.7)$$

The group of induced canonical transformations of  $T^*\mathbb{R}^n$  to itself is given by

$$\Phi_A^*(\mathbf{q}, \mathbf{p}) = (A^{-1}\mathbf{q}, A^T\mathbf{p}), \quad (6.4.8)$$

which is readily verified. Notice that this reduces to the same transformation of  $\mathbf{q}$  and  $\mathbf{p}$  when  $A$  is orthogonal.  $\blacklozenge$

### Exercises

- ◇ **Exercise 6.4-1.** Let the multiplicative group  $\mathbb{R} \setminus \{0\}$  act on  $\mathbb{R}^n$  by  $\Phi_\lambda(\mathbf{q}) = \lambda\mathbf{q}$ . Calculate the cotangent lift of this action.

## 6.5 Generating Functions

Consider a symplectic diffeomorphism  $\varphi : T^*Q_1 \rightarrow T^*Q_2$  described by functions

$$p_i = p_i(q^j, s^j), \quad r_i = r_i(q^j, s^j), \quad (6.5.1)$$

where  $(q^i, p_i)$  and  $(s^j, r_j)$  are cotangent coordinates on  $T^*Q_1$  and on  $T^*Q_2$ , respectively. In other words, assume that we have a map

$$\Gamma : Q_1 \times Q_2 \rightarrow T^*Q_1 \times T^*Q_2 \quad (6.5.2)$$

whose image is the graph of  $\varphi$ . Let  $\Theta_1$  be the canonical one-form on  $T^*Q_1$  and  $\Theta_2$  be that on  $T^*Q_2$ . By definition,

$$\mathbf{d}(\Theta_1 - \varphi^*\Theta_2) = 0. \quad (6.5.3)$$

This implies, in view of (6.5.1), that

$$p_i dq^i - r_i ds^i \quad (6.5.4)$$

is closed. Restated,  $\Gamma^*(\Theta_1 - \Theta_2)$  is closed. This holds if (and implies locally by the Poincaré lemma)

$$\Gamma^*(\Theta_1 - \Theta_2) = \mathbf{d}S \quad (6.5.5)$$

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for a function  $S(q, s)$ . In coordinates, (6.5.5) reads

$$p_i dq^i - r_i ds^i = \frac{\partial S}{\partial q^i} dq^i + \frac{\partial S}{\partial s^i} ds^i \quad (6.5.6)$$

which is equivalent to

$$p_i = \frac{\partial S}{\partial q^i}, \quad r_i = -\frac{\partial S}{\partial s^i}. \quad (6.5.7)$$

One calls  $S$  a **generating function** for the canonical transformation. With generating functions of this sort, one may run into singularities even with the identity map! See Exercise **6.5-1**.

Presupposed relations other than (6.5.1) lead to different conclusions than (6.5.7). Point transformations are generated in this sense; if  $S(q^i, r_j) = s^j(q)r_j$ , then

$$s^i = \frac{\partial S}{\partial r_i} \quad \text{and} \quad p_i = \frac{\partial S}{\partial q^i}. \quad (6.5.8)$$

(Here one writes  $p_i dq^i + s^i dr_i = \mathbf{d}S$ .)

In general, consider a diffeomorphism  $\varphi : P_1 \rightarrow P_2$  of one symplectic manifold  $(P_1, \Omega_1)$  to another  $(P_2, \Omega_2)$  and denote the graph of  $\varphi$ , by  $\Gamma(\varphi) \subset P_1 \times P_2$ . Let  $i_\varphi : \Gamma(\varphi) \rightarrow P_1 \times P_2$  be the inclusion and let  $\Omega = \pi_1^* \Omega_1 - \pi_2^* \Omega_2$ , where  $\pi_i : P_1 \times P_2 \rightarrow P_i$  is the projection. One verifies that  $\varphi$  is symplectic if and only if  $i_\varphi^* \Omega = 0$ . (Indeed, since  $\pi_1 \circ i_\varphi$  is the projection restricted to  $\Gamma(\varphi)$  and  $\pi_2 \circ i_\varphi = \varphi \circ \pi_1$  on  $\Gamma(\varphi)$ , it follows that

$$i_\varphi^* \Omega = (\pi_1|_{\Gamma(\varphi)})^*(\Omega_1 - \varphi^* \Omega_2),$$

and hence  $i_\varphi^* \Omega = 0$  iff  $\varphi$  is symplectic because  $(\pi_1|_{\Gamma(\varphi)})^*$  is injective.) In this case, one says  $\Gamma(\varphi)$  is an **isotropic** submanifold of  $P_1 \times P_2$  (equipped with the symplectic form  $\Omega$ ); in fact, since  $\Gamma(\varphi)$  has half the dimension of  $P_1 \times P_2$ , it is **maximally isotropic**, or a **Lagrangian manifold**.

Now suppose one chooses a form  $\Theta$  such that  $\Omega = -\mathbf{d}\Theta$ . Then  $i_\varphi^* \Omega = -\mathbf{d}i_\varphi^* \Theta = 0$ , so *locally* on  $\Gamma(\varphi)$  there is a function  $S : \Gamma(\varphi) \rightarrow \mathbb{R}$  such that

$$i_\varphi^* \Theta = \mathbf{d}S. \quad (6.5.9)$$

This defines the **generating function** of the canonical transformation  $\varphi$ . Since  $\Gamma(\varphi)$  is diffeomorphic to  $P_1$  and also to  $P_2$  we can regard  $S$  as a function on  $P_1$  or  $P_2$ . If  $P_1 = T^*Q_1$  and  $P_2 = T^*Q_2$ , we can equally well regard (at least locally)  $S$  as defined on  $Q_1 \times Q_2$ . In this way, the general construction of generating functions reduces to the case in equations (6.5.7) and (6.5.8) above. By making other choices of  $Q$ , the reader can construct other generating functions and reproduce formulas in, for instance, Goldstein [1980] or Whittaker [1927]. The approach here is based on Sniatycki and Tulczyjew [1971].

Generating functions play an important role in Hamilton-Jacobi theory, in the quantum-classical mechanical relationship (where  $S$  plays the role of the quantum mechanical phase), and in numerical integration schemes for Hamiltonian systems. We shall see a few of these aspects later on.

### Exercises

- ◇ **Exercise 6.5-1.** Show that

$$S(q^i, s^j, t) = \frac{1}{2t} \|\mathbf{q} - \mathbf{s}\|^2$$

generates a canonical transformation that is the identity at  $t = 0$ .

- ◇ **Exercise 6.5-2.** (A first-order symplectic integrator). Given  $H$ , let

$$S(q^i, r_j, t) = r_k q^k - tH(q^i, r_j).$$

Show that  $S$  generates a canonical transformation which is a first-order approximation to the flow of  $X_H$  for small  $t$ .

## 6.6 Fiber Translations and Magnetic Terms

**Momentum Shifts.** We saw above that cotangent lifts provide a basic construction of canonical transformations. Fiber translations provide a second.

**Proposition 6.6.1 (Momentum Shifting Lemma).** *Let  $A$  be a one-form on  $Q$  and let  $t_A : T^*Q \rightarrow T^*Q$  be defined by  $\alpha_q \mapsto \alpha_q + A(q)$ , where  $\alpha_q \in T_q^*Q$ . Let  $\Theta$  be the canonical one-form on  $T^*Q$ . Then*

$$t_A^* \Theta = \Theta + \pi_Q^* A, \quad (6.6.1)$$

where  $\pi_Q : T^*Q \rightarrow Q$  is the projection. Hence

$$t_A^* \Omega = \Omega - \pi_Q^* \mathbf{d}A, \quad (6.6.2)$$

where  $\Omega = -\mathbf{d}\Theta$  is the canonical symplectic form. Thus,  $t_A$  is a canonical transformation if and only if  $\mathbf{d}A = 0$ .

**Proof.** We prove this using a finite-dimensional coordinate computation. The reader is asked to supply the coordinate-free and infinite-dimensional proofs as an exercise. In coordinates,  $t_A$  is the map

$$t_A(q^i, p_j) = (q^i, p_j + A_j). \quad (6.6.3)$$

Thus,

$$t_A^* \Theta = t_A^*(p_i \mathbf{d}q^i) = (p_i + A_i) \mathbf{d}q^i = p_i \mathbf{d}q^i + A_i \mathbf{d}q^i, \quad (6.6.4)$$

which is the coordinate expression for  $\Theta + \pi_Q^* A$ . The remaining assertions follow directly from this. ■

In particular, fiber translation by the differential of a function  $A = \mathbf{d}f$  is a canonical transformation; in fact,  $f$  induces, in the sense of the preceding section, a generating function (see Exercise 6.6-2). The two basic classes of canonical transformations, lifts, and fiber translations, play an important part in mechanics.

**Magnetic Terms.** A symplectic form on  $T^*Q$ , different from the canonical one, is obtained in the following way. Let  $B$  be a closed two-form on  $Q$ . Then  $\Omega - \pi_Q^*B$  is a closed two-form on  $T^*Q$ , where  $\Omega$  is the canonical two-form. To see that  $\Omega - \pi_Q^*B$  is (weakly) nondegenerate, use the fact that in a local chart this form is given at the point  $(w, \alpha)$  by

$$((u, \beta), (v, \gamma)) \mapsto \langle \gamma, u \rangle - \langle \beta, v \rangle - B(w)(u, v). \quad (6.6.5)$$

**Proposition 6.6.2.**

- (i) Let  $\Omega$  be the canonical two-form on  $T^*Q$  and let  $\pi_Q : T^*Q \rightarrow Q$  be the projection. If  $B$  is a closed two-form on  $Q$ , then

$$\Omega_B = \Omega - \pi_Q^*B \quad (6.6.6)$$

is a (weak) symplectic form on  $T^*Q$ .

- (ii) Let  $B$  and  $B'$  be closed two-forms on  $Q$  and assume that  $B - B' = \mathbf{d}A$ . Then the mapping  $t_A$  (fiber translation by  $A$ ) is a symplectic diffeomorphism of  $(T^*Q, \Omega - \pi_Q^*B)$  with  $(T^*Q, \Omega - \pi_Q^*B')$ .

**Proof.** Part (i) follows from the momentum shifting lemma. For (ii), use formula (6.6.2) to get

$$t_A^*\Omega = \Omega - \pi_Q^*\mathbf{d}A = \Omega - \pi_Q^*B + \pi_Q^*B', \quad (6.6.7)$$

so that

$$t_A^*(\Omega - \pi_Q^*B') = \Omega - \pi_Q^*B$$

since  $\pi_Q \circ t_A = \pi_Q$ . ■

Symplectic forms of the type  $\Omega_B$  arise in the reduction process.<sup>2</sup> In the following section, we explain why the extra term  $\pi_Q^*B$  is called a **magnetic term**.

## Exercises

- ◇ **Exercise 6.6-1.** Provide the intrinsic proof of Proposition 6.6.1.
- ◇ **Exercise 6.6-2.** If  $A = \mathbf{d}f$ , use a coordinate calculation to check that  $S(q^i, r_i) = r_i q^i - f(q^i)$  is a generating function for  $t_A$ .

<sup>2</sup>Magnetic terms come up in what is called the **cotangent bundle reduction theorem**; see Smale [1972], Abraham and Marsden [1978], Kummer [1981], Nill [1983], Montgomery, Marsden, and Ratiu [1984], Gozzi and Thacker [1987], and Marsden [1992].

## 6.7 A Particle in a Magnetic Field

Let  $B$  be a closed two-form on  $\mathbb{R}^3$  and let  $\mathbf{B} = B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}$  be the associated divergence-free vector field, that is,

$$\mathbf{i}_B(dx \wedge dy \wedge dz) = B,$$

so that

$$B = B_x dy \wedge dz - B_y dx \wedge dz + B_z dx \wedge dy.$$

Thinking of  $\mathbf{B}$  as a magnetic field, the equations of motion for a particle with charge  $e$  and mass  $m$  are given by the **Lorentz force law**:

$$m \frac{d\mathbf{v}}{dt} = \frac{e}{c} \mathbf{v} \times \mathbf{B}, \quad (6.7.1)$$

where  $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z})$ . On  $\mathbb{R}^3 \times \mathbb{R}^3$ , that is,  $(\mathbf{x}, \mathbf{v})$ -space, consider the symplectic form

$$\Omega_B = m(dx \wedge d\dot{x} + dy \wedge d\dot{y} + dz \wedge d\dot{z}) - \frac{e}{c} B \quad (6.7.2)$$

that is, (6.6.6). As Hamiltonian, take the kinetic energy:

$$H = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad (6.7.3)$$

writing  $X_H(u, v, w) = (u, v, w, \dot{u}, \dot{v}, \dot{w})$ , the condition

$$\mathbf{d}H = \mathbf{i}_{X_H} \Omega_B \quad (6.7.4)$$

is

$$\begin{aligned} m(\dot{x} d\dot{x} + \dot{y} d\dot{y} + \dot{z} d\dot{z}) &= m(u d\dot{x} - \dot{u} dx + v d\dot{y} - \dot{v} dy \\ &\quad + w d\dot{z} - \dot{w} dz) - \frac{e}{c} [B_x v dz - B_x w dy \\ &\quad - B_y u dz + B_y w dx + B_z u dy - B_z v dx], \end{aligned}$$

which is equivalent to

$$\begin{aligned} u &= \dot{x}, & v &= \dot{y}, & w &= \dot{z}, \\ m\dot{u} &= \frac{e}{c}(B_z v - B_y w), \\ m\dot{v} &= \frac{e}{c}(B_x w - B_z u), \\ m\dot{w} &= \frac{e}{c}(B_y u - B_x v), \end{aligned}$$

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that is, to

$$\left. \begin{aligned} m\ddot{x} &= \frac{e}{c}(B_z\dot{y} - B_y\dot{z}), \\ m\ddot{y} &= \frac{e}{c}(B_x\dot{z} - B_z\dot{x}), \\ m\ddot{z} &= \frac{e}{c}(B_y\dot{x} - B_x\dot{y}), \end{aligned} \right\} \quad (6.7.5)$$

which is the same as (6.7.1). Thus *the equations of motion for a particle in a magnetic field are Hamiltonian, with energy equal to the kinetic energy and with the symplectic form  $\Omega_B$ .*

If  $B = \mathbf{d}A$ ; that is,  $\mathbf{B} = \nabla \times \mathbf{A}$ , where  $\mathbf{A}^b = A$ , then the map  $t_A : (\mathbf{x}, \mathbf{v}) \mapsto (\mathbf{x}, \mathbf{p})$ , where  $\mathbf{p} = m\mathbf{v} + e\mathbf{A}/c$  pulls back the canonical form to  $\Omega_B$  by the momentum shifting lemma. Thus, equations (6.7.1) are also Hamiltonian relative to the canonical bracket on  $(\mathbf{x}, \mathbf{p})$ -space with the Hamiltonian

$$H_A = \frac{1}{2m} \left\| \mathbf{p} - \frac{e}{c} \mathbf{A} \right\|^2. \quad (6.7.6)$$

#### Remarks.

1. Not every magnetic field can be written as  $\mathbf{B} = \nabla \times \mathbf{A}$  on Euclidean space. For example, the **field of a magnetic monopole of strength  $g \neq 0$** , namely

$$\mathbf{B}(\mathbf{r}) = g \frac{\mathbf{r}}{\|\mathbf{r}\|^3}, \quad (6.7.7)$$

cannot be written this way since the flux of  $\mathbf{B}$  through the unit sphere is  $4\pi g$ , yet Stokes' theorem applied to the two-sphere would give zero; see Exercise 4.4-3. Thus, one might think that the Hamiltonian formulation involving only  $\mathbf{B}$  (that is, using  $\Omega_B$  and  $H$ ) is preferable. However, there is a way to recover the magnetic potential  $\mathbf{A}$  by regarding it as a connection on a *nontrivial bundle* over  $\mathbb{R}^3 \setminus \{0\}$ . (This bundle over the sphere  $S^2$  is the **Hopf fibration**  $S^3 \rightarrow S^2$ .) This same construction can be carried out using reduction and we shall do so later. For a readable account of some aspects of this situation, see Yang [1985].

2. When one studies the motion of a particle in a Yang–Mills field, one finds a beautiful generalization of this construction and related ideas using the theory of principal bundles; see Sternberg [1977], Weinstein [1978], and Montgomery [1984].

3. In Chapter 8 we study centrifugal and Coriolis forces and will see some structures analogous to those here. ♦

### Exercises

- ◇ **Exercise 6.7-1.** Show that particles in constant magnetic fields move in helices.
- ◇ **Exercise 6.7-2.** Verify “by hand” that  $\frac{1}{2}m\|\mathbf{v}\|^2$  is conserved for a particle moving in a magnetic field.

## 6.8 Linearization of Hamiltonian Systems

This optional section may be downloaded from the world wide web at:  
<http://www.cds.caltech.edu/~marsden>